Regularity of the sample paths of a class of second order spde's

by

Robert C. Dalang [(*)]
Institut de Mathématiques
Ecole Polytechnique Fédérale
1015 Lausanne
Switzerland
e-mail: robert.dalang@epfl.ch

and

Marta Sanz-Solé [(**)]
Facultat de Matemàtiques
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Spain
e-mail: marta.sanz@ub.edu

Abstract: We study the sample path regularity of the solutions of a class of spde's which are second order in time and that includes the stochastic wave equation. Non-integer powers of the spatial Laplacian are allowed. The driving noise is white in time and spatially homogeneous. Continuing with the work initiated in Dalang and Mueller (2003), we prove that the solutions belong to a fractional L^2-Sobolev space. We also prove Hölder continuity in time and therefore, we obtain joint Hölder continuity in the time and space variables. Our conclusions rely on a precise analysis of the properties of the stochastic integral used in the rigourous formulation of the spde, as introduced by Dalang and Mueller. For spatial covariances given by Riesz kernels, we show that our results are optimal.

Key words and phrases. Stochastic partial differential equations, path regularity, spatially homogeneous random noise, wave equation, fractional Laplacian.

MSC 2000 Subject Classifications. Primary 60H15; Secondary 35R60, 35L05.

(*) Partially supported by the Swiss National Foundation for Scientific Research.

(**) Partially supported by the grant BFM2003-01345 from *Dirección General de Investigación, Ministerio de Ciencia y Tecnología*, Spain.

1 Introduction

Modelling with stochastic partial differential equations (spde's) provides successful understanding of the evolution of many physical phenomena. A basic issue is how to choose the ingredients so that the spde possesses a solution in a strong sense—giving rise to a function-valued stochastic process—and to fix, in the most precise way possible, the function space that contains the sample paths of the solution. It is well known that this amounts to finding the right balance between the roughness of the driving noise—the stochastic input in the model—and the singularities of the differential operator that defines the equation, which may depend on the dimension.

This paper focusses on the analysis of the following class of spde's:

$$\left(\frac{\partial^2}{\partial t^2} + (-\Delta)^{(k)}\right) u(t,x) = \sigma\big(u(t,x)\big)\dot{F}(t,x) + b\big(u(t,x)\big),$$
$$u(0,x) = v_0(x), \qquad \frac{\partial}{\partial t}u(0,x) = \tilde{v}_0(x). \tag{1}$$

In this equation, $t \in [0,T]$ for some fixed $T > 0$, $x \in \mathbb{R}^d$, $d \in \mathbb{N}$, $k \in]0,\infty[$ and $\Delta^{(k)}$ denotes the fractional Laplacian on $\mathbb{R}^d$. This includes for instance the stochastic wave equation in any spatial dimension d. The coefficients σ and b are Lipschitz continuous functions and satisfy $|\sigma(z)| + |b(z)| \le C|z|$, for some positive constant C. The generalized process $\dot{F}$ is a Gaussian random field, white in time and spatially homogeneous with spatial correlation. More precisely, let Γ be a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the space of Schwartz test functions (see [18]). On a probability space $(\Omega, \mathcal{F}, P)$, we define a Gaussian process $F = \big(F(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\big)$ with mean zero and covariance functional given by

$$E\big(F(\varphi)F(\psi)\big) = \int_{\mathbb{R}_+} ds \int_{\mathbb{R}^d} \Gamma(dx)(\varphi(s) * \tilde{\psi}(s))(x),$$

where $\tilde{\psi}(s)(x) = \psi(s)(-x)$.

Using an extension of Walsh's stochastic integral with respect to martingale measures [22], developed in [5], we give a rigourous meaning to problem (1) in a mild form (see Equation (39)). In fact, in [5] a particular case of Equation (1) (when $k \in \mathbb{N}$ is an integer and $b \equiv 0$) was introduced and studied.

Let $\mu = \mathcal{F}^{-1}\Gamma$ be the spectral measure of F and assume that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^k} < \infty. \tag{2}$$

Under suitable assumptions on the initial condition and the restrictions on k and b mentioned above, Theorem 9 in [5] establishes the existence of a

unique solution satisfying

$$\sup_{0\le t\le T} E(\|u(t)\|^2_{L^2(\mathbb{R}^d)}) < \infty,$$

for which $t \mapsto u(t) \in L^2(\mathbb{R}^d)$ is mean-square continuous.

Here, we want to study the regularity properties of the sample paths—both in time and in space—of Equation (1), when conditions stronger than condition (2) are imposed.

In several examples of spde's driven by spatially homogeneous noise, one can prove that their solutions are real-valued random fields $u = (u(t,x), (t,x) \in [0,T] \times \mathbb{R}^d)$. This is the case for the stochastic heat equation in any spatial dimension $d \ge 1$, for the stochastic wave equation in dimension $d \in \{1,2\}$, or even in dimension $d = 3$, if the initial conditions vanish ([3], [4], [13]). Joint Hölder continuity in (t,x) of the sample paths of the solution can usually be obtained using Kolmogorov's continuity condition ([13], [16]). However, for more general equations, such as those considered in this paper, one can only expect solutions $u = (u(t),\ t \in [0,T])$ taking values in some function space. Kolmogorov's condition is still well suited for establishing regularity properties in time, but it is not for the study of spatial continuity. Regularity in space may be obtained by means of Sobolev type imbeddings, if one could prove that the solution takes values in some fractional Sobolev space H^α_p, $p \in [1,\infty[$, $\alpha \in [0,\infty[$. Indeed, H^α_p is imbedded in the space of γ-Hölder continuous functions $\mathcal{C}^\gamma(\mathbb{R}^d)$, for any $\gamma \in]0, \alpha - \frac{d}{p}[$, whenever $\alpha > \frac{d}{p}$. This fact explains one of the main advantages of an L^p-theory for spde's, for arbitrary values of p, leading to optimal results in γ.

Until now, L^p-theory for spde's has been mainly developed for parabolic spde's (see for instance [11] and the references herein). Recently, we have been able to use an L^p approach to study the sample path behaviour in (t,x) of the stochastic wave equation in dimension $d = 3$ (see [6]), driven by the type of noise described above and with a covariance function whose singularity is given by a Riesz kernel. The methods used in the analysis of this particular equation are very much related to the special form of the fundamental solution of the equation and of the covariance function of the noise; they do not seem to be exportable to the more general situation we are considering here.

In this paper, we establish sufficient conditions on the spectral measure μ of the noise that ensure that the solution of Equation (1) belongs a.s. to some fractional Sobolev space H^α_2, for some $\alpha \in [0, k[$. Then we prove Hölder continuity in time of the solution and show that the results are optimal when the covariance measure is a Riesz kernel.

Let M be the martingale measure extension of the process F, obtained in [3] (see also [4]) and let Z be an $L^2(\mathbb{R}^d)$-valued stochastic process. In Section 2, we prove that, under suitable assumptions on the $\mathcal{S}'(\mathbb{R}^d)$-valued

function G, the stochastic integral

$$v_{G,Z}(T) = \int_0^T \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s,y) M(ds, dy)$$

introduced in [5], which defines a random element of $L^2(\mathbb{R}^d)$, belongs in fact to $H_2^\alpha(\mathbb{R}^d)$ and is such that $E\big(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) < \infty$. To establish this fact, we will need to prove the existence of the Fourier transform of the stochastic integral $v_{G,Z}(T)$. Recall [21] that for a function $g \in H_2^\alpha(\mathbb{R}^d)$,

$$\|g\|^2_{H_2^\alpha(\mathbb{R}^d)} = \int_{\mathbb{R}^d} d\xi \, (1 + |\xi|^2)^\alpha \, |\mathcal{F}g(\xi)|^2,$$

where, for $\varphi \in C_0^\infty(\mathbb{R}^d)$,

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} dx \, e^{i\xi x} \, \varphi(x).$$

Let $\mathcal{L} = \partial^2_{tt} + (-\Delta)^{(k)}$, $k \in]0, \infty[$. We prove that, if for some $\alpha \in [0, k[$,

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^{k-\alpha}} < \infty, \tag{3}$$

then the preceding result applies to the fundamental solution of $\mathcal{L}u = 0$. For the other results of this paper, we will assume property (3). We note that we treat indifferently the case of integer and fractional powers of the Laplacian.

Section 2 is devoted to studying path properties in time of the stochastic integral

$$v_{G,Z}(t) = \int_0^t \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s,y) M(ds, dy)$$

and the Hölder continuity of

$$u_{G,Z}(t) = \int_0^t \int_{\mathbb{R}^d} G(t-s, \cdot - y) Z(s,y) M(ds, dy). \tag{4}$$

We first identify the increasing process of the $H_2^\alpha(\mathbb{R}^d)$-valued martingale $\big(v_{G,Z}(t), t \in [0,T]\big)$. Fix $\alpha \in [0,k[$ and assume that there exists $\eta \in]\frac{\alpha}{d}, 1[$ such that the following condition, which is stronger than (3), holds:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1 + |\xi|^2)^{k\eta-\alpha}} < \infty. \tag{5}$$

Using Kolmogorov's continuity condition, we obtain that the sample paths of $(u_{G,Z}(t),\ t \in [0,T])$ are a.s. Hölder continuous. In the particular case where $\Gamma(dx) = |x|^{-\beta}$, $\beta \in]0, d[$, the results are proved to be optimal. By means of the Sobolev imbedding theorem, we also obtain Hölder continuity

in the space variable. However, the conditions for validity of this result are rather restrictive.

In Section 3, we transfer the results of the preceding sections to the solution of Equation (1). Fix $\alpha \in [0, k[$, assume (3) and that the initial conditions are such that $v_0 \in H_2^\alpha(\mathbb{R}^d)$ and $\tilde{v}_0 \in H_2^{\alpha-k}(\mathbb{R}^d)$. We prove the existence of a solution to (1) satisfying

$$\sup_{0\le t\le T} E\big(\|u(t)\|^q_{H_2^\alpha(\mathbb{R}^d)}\big) < \infty.$$

Replacing assumption (3) by (5) and under additional (but natural) hypotheses on the initial conditions, we obtain Hölder continuity in time of the solution of the equation.

2 The stochastic integral as a random vector with values in a fractional Sobolev space

In this section, we consider the stochastic integral defined in Theorem 6 of [5]. Our aim is to prove that under suitable assumptions, this integral takes its values in the fractional Sobolev space $H_2^\alpha(\mathbb{R}^d)$, for some $\alpha \in]0, \infty[$.

Throughout this section, let $\mathcal{F}_s$ be the σ-field generated by the martingale measure $(M_t,\ 0 \le t \le s)$ described in the introduction. We consider a stochastic process $Z = (Z(s),\ s \in [0,T])$ with values in $L^2(\mathbb{R}^d)$ such that $Z(s)$ is $\mathcal{F}_s$-measurable and the mapping $s \mapsto Z(s)$ is mean-square continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$.

The main result of this section is as follows.

Theorem 1 *Consider a deterministic map $G : [0,T] \longrightarrow \mathcal{S}'(\mathbb{R}^d)$. Fix $\alpha \in [0, \infty[$ and assume that the following three conditions hold:*

(i) *For each $s \in [0,T]$, $\mathcal{F}G(s)$ is a function and*

$$\sup_{0\le s\le T} \sup_{\xi\in\mathbb{R}^d} (1+|\xi|^2)^{\frac{\alpha}{2}} \big|\mathcal{F}G(s)(\xi)\big| < \infty.$$

(ii) *For all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

$$\sup_{0\le s\le T} \sup_{x\in\mathbb{R}^d} \big|(G(s) * \psi)(x)\big| < \infty.$$

(iii)

$$\int_0^T ds \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \big|\mathcal{F}G(s)(\xi-\eta)\big|^2 < \infty.$$

Then the stochastic integral

$$v_{G,Z}(T) = \int_0^T \int_{\mathbb{R}^d} G(s, \cdot - y) Z(s,y) M(ds,dy)$$

satisfies

$$E(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}) < \infty$$

and

$$\begin{aligned} E(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}) &= E(\|v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T)\|^2_{L^2(\mathbb{R}^d)}) \\ &= I^\alpha_{G,Z}, \end{aligned} \tag{6}$$

where

$$v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T) = \int_0^T \int_{\mathbb{R}^d} (I-\Delta)^{\frac{\alpha}{2}} G(s, \cdot - y) Z(s,y) M(ds,dy)$$

and

$$\begin{aligned} I^\alpha_{G,Z} = \int_0^T ds \int_{\mathbb{R}^d} d\xi E(|\mathcal{F}Z(s)(\xi)|^2) \\ \times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(s)(\xi-\eta)\right|^2. \end{aligned} \tag{7}$$

The proof of this theorem relies on a preliminary result that identifies the Fourier transform of the stochastic integral $v_{G,Z}$ for G and Z satisfying more restrictive assumptions than those above, namely:

(G1') For each $s \in [0,T]$, $G(s) \in \mathcal{C}^\infty(\mathbb{R}^d)$, $\mathcal{F}G(s)$ is a function,

$$\sup_{0\le s\le T} \sup_{x\in\mathbb{R}^d} |G(s,x)| < \infty \qquad \text{and} \qquad \sup_{0\le s\le T} \sup_{\xi\in\mathbb{R}^d} |\mathcal{F}G(s)(\xi)| < \infty.$$

(G2) For $s \in [0,T]$, $Z(s) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ a.s., and there is a compact set $K \subset \mathbb{R}^d$ such that $\operatorname{supp} Z(s) \subset K$, for $s \in [0,T]$. In addition, the mapping $s \mapsto Z(s)$ is mean-square continuous from $[0,T]$ into $L^2(\mathbb{R}^d)$.

(G3) $I_{G,Z} < \infty$, where

$$I_{G,Z} = \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s)(\xi)|^2) \int_{\mathbb{R}^d} \mu(d\eta) |\mathcal{F}G(s)(\xi-\eta)|^2. \tag{8}$$

Notice that our assumption **(G1')** is stronger than **(G1)** in [5] (which does not suppose the boundedness of the Fourier transform of G), while **(G2)** and **(G3)** appear with the same name in [5].

Under **(G1')**, **(G2)**, and **(G3)**, the stochastic integral

$$v_{G,Z}(T)(x) = \int_0^T \int_{\mathbb{R}^d} G(s, x-y) Z(s,y) M(ds, dy)$$

is well-defined, for any $x \in \mathbb{R}^d$, as a Walsh stochastic integral (see Lemma 1 in [5]). The integral

$$\int_0^T \int_{\mathbb{R}^d} \mathcal{F}G(s, \cdot - y)(\xi) Z(s,y) M(ds, dy)$$

is also well-defined as a Walsh stochastic integral. Indeed, $\mathcal{F}G(s)(\cdot - y)(\xi) = e^{i\xi\cdot y}\mathcal{F}G(s)(\xi)$, and

$$\begin{aligned}
&E\Big(\int_0^T ds \int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz |e^{i\xi\cdot z}\mathcal{F}G(s)(\xi) Z(s,z) e^{i\xi(y-z)}\mathcal{F}G(s)(\xi) Z(s, y-z)|\Big) \\
&\qquad \leq \sup_{0\leq s\leq T} \sup_{\xi\in\mathbb{R}^d} |\mathcal{F}G(s)(\xi)|^2 \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dy) E\big(|Z(s,\cdot)| * |\tilde{Z}(s,\cdot)|\big)(y) \\
&\qquad \leq C \int_0^T ds\, E(\|Z(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}) \Gamma(K-K) < \infty.
\end{aligned}$$

Proposition 1 *We assume the hypotheses* (**G1′**), (**G2**) *and* (**G3**). *Then the Fourier transform* $\mathcal{F}v_{G,Z}(T)$ *of the stochastic integral* $v_{G,Z}(T)$ *is given by*

$$\mathcal{F}v_{G,Z}(T)(\xi) = \int_0^T \int_{\mathbb{R}^d} \mathcal{F}G(s, \cdot - y)(\xi) Z(s,y) M(ds, dy).$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. We want to check that

$$\langle v_{G,Z}(T), \mathcal{F}^{-1}\varphi\rangle = \langle \int_0^T \int_{\mathbb{R}^d} \mathcal{F}G(s, \cdot - y)(\cdot) Z(s,y) M(ds, dy), \varphi\rangle, \tag{9}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$.

We verify the assumptions of the stochastic Fubini's theorem in [22]: since G is uniformly bounded and $Z(s)$ has compact support,

$$\begin{aligned}
&E\Big(\int_{\mathbb{R}^d} dx \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz\, |\mathcal{F}^{-1}\varphi(x)|^2 |G(s, x-z)| \\
&\qquad\qquad \times |Z(s,z)|\, |G(s, x-z+y)|\, |Z(s, z-y)|\Big) \\
&\leq CE\Big(\int_{\mathbb{R}^d} dx\, |\mathcal{F}^{-1}\varphi(x)|^2 \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz\, |Z(s,z)|\, |\tilde{Z}(s, y-z)|\Big) \\
&\leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)} \Gamma(K-K) \int_0^T ds\, E(\|Z(s)\|^2_{L^2(\mathbb{R}^d)}) < \infty.
\end{aligned}$$

Applying this Fubini's theorem and Plancherel's identity, we obtain

$$\begin{aligned}\langle v_{G,Z}(T), \mathcal{F}^{-1}\varphi\rangle &= \int_{\mathbb{R}^d} dx \, \Big(\int_0^T \int_{\mathbb{R}^d} \mathcal{F}^{-1}\varphi(x) G(s, x-y) Z(s,y) M(ds,dy)\Big) \\ &= \int_0^T \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} dx \, \mathcal{F}^{-1}\varphi(x) G(s,x-y)\Big) Z(s,y) M(ds,dy)\Big) \\ &= \int_0^T \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} d\xi \, \varphi(\xi) \mathcal{F}G(s)(\xi) \exp(iy\cdot\xi)\Big) Z(s,y) M(ds,dy). \end{aligned} \tag{10}$$

In order to apply again the stochastic Fubini's theorem, we note that

$$\begin{aligned} &E\Big(\int_{\mathbb{R}^d} d\xi \int_0^T ds \int_{\mathbb{R}^d} \Gamma(dy) \int_{\mathbb{R}^d} dz \, |\varphi(\xi)|^2 |\mathcal{F}G(s)(\xi)|^2 |Z(s,z)| \, |Z(s,y-z)| \Big) \\ &\leq \Gamma(K-K) \int_{\mathbb{R}^d} d\xi \int_0^T ds \, |\varphi(\xi)|^2 |\mathcal{F}G(s)(\xi)|^2 E(\|Z(s)\|^2_{L^2(\mathbb{R}^d)}) \\ &\leq C \|\varphi\|^2_{L^2(\mathbb{R}^d)} \sup_{0\leq s\leq T} E(\|Z(s)\|^2_{L^2(\mathbb{R}^d)}) \sup_{0\leq s\leq T} \sup_{\xi\in\mathbb{R}^d} |\mathcal{F}G(s)(\xi)|^2. \end{aligned} \tag{11}$$

Therefore, applying again the above-mentioned Fubini's theorem shows that the last right-hand side of (11) is equal to

$$\int_{\mathbb{R}^d} d\xi \, \varphi(\xi) \Big(\int_0^T \int_{\mathbb{R}^d} \mathcal{F}G(s, \cdot - y)(\xi) Z(s,y) M(ds,dy)\Big),$$

which establishes (9). □

Proof of Theorem 1. We proceed in several steps.

Step 1. Assume first that G and Z satisfy the assumptions **(G1')**, **(G2)** and **(G3)**. Suppose also that $G^\alpha(s) := (I-\Delta)^{\frac{\alpha}{2}} G(s)$ satisfies **(G1')** and that $I^\alpha_{G,Z} < \infty$ (this last condition is implied by (iii)). Then the stochastic integral $v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T)$ is well-defined in Walsh's sense and satisfies

$$E(\|v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T)\|^2_{L^2(\mathbb{R}^d)}) = I^\alpha_{G,Z}. \tag{12}$$

Indeed, this follows from Lemma 1 in [5]. Moreover, Proposition 1 implies that

$$\mathcal{F} v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T) = v_{\mathcal{F}(I-\Delta)^{\frac{\alpha}{2}} G,Z}(T). \tag{13}$$

By the definition of the norm in $H^\alpha_2(\mathbb{R}^d)$, Plancherel's theorem and

Proposition 1, we obtain

$$
\begin{aligned}
&E\big(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) = E\Big(\int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^\alpha \big|\mathcal{F}v_{G,Z}(T)(\xi)\big|^2\Big)\\
&= E\Big(\int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^\alpha \big|v_{\mathcal{F}G,Z}(T)(\xi)\big|^2\Big)\\
&= E\Big(\int_{\mathbb{R}^d} d\xi\,\Big|\int_0^T\int_{\mathbb{R}^d}(1+|\xi|^2)^{\frac{\alpha}{2}}\mathcal{F}G(s,\cdot-y)(\xi)Z(s,y)M(ds,dy)\Big|^2\Big)\\
&= E\Big(\int_{\mathbb{R}^d} d\xi\,\Big|\int_0^T\int_{\mathbb{R}^d}\mathcal{F}\big((I-\Delta)^{\frac{\alpha}{2}}G(s,\cdot-y)\big)(\xi)Z(s,y)M(ds,dy)\Big|^2\Big)\\
&= E\big(\|v_{(I-\Delta)^{\frac{\alpha}{2}}G,Z}(T)\|^2_{L^2(\mathbb{R}^d)}\big).
\end{aligned}
\tag{14}
$$

Consequently, the theorem is proved in this particular situation. Notice that (6) follows from (12) and (14).

Step 2. Assume that $G(s)$ and $G^\alpha(s)$ satisfy **(G1')** and that condition (iii) holds. By Lemma 3 in [5], there exists a sequence of stochastic processes $(Z_n, n \geq 1)$ satisfying **(G2)** and **(G3)** such that $\lim_{n\to\infty} I^0_{G^\alpha,Z_n-Z} = 0$ and

$$v_{G^\alpha,Z}(T) = \lim_{n\to\infty} v_{G^\alpha,Z_n}(T),$$

with the limit taken in $L^2(\Omega; L^2(\mathbb{R}^d))$. The properties of G^α ensure that $\lim_{n\to\infty} I^\alpha_{G,Z-Z_n} = 0$ as well.

We want to prove that $(v_{G,Z_n}(T),\ n \geq 1)$ is a Cauchy sequence in $L^2(\Omega; H_2^\alpha(\mathbb{R}^d))$. Since $H_2^\alpha(\mathbb{R}^d))$ is imbedded in $L^2(\mathbb{R}^d)$, the two limits of the sequence—in $L^2(\Omega; H_2^\alpha(\mathbb{R}^d))$ and in $L^2(\Omega; L^2(\mathbb{R}^d))$—must coincide.

By the results proved in Step 1,

$$\lim_{n,m\to\infty} E\big(\|v_{G,Z_n-Z_m}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) = \lim_{n,m\to\infty} I^\alpha_{G,Z_n-Z_m} = 0. \tag{15}$$

Let us now prove (6) in this particular case. The previous convergence, the results stated in the first part of the proof and Lemma 3 in [5] applied to $g := (I-\Delta)^{\frac{\alpha}{2}}G$ yield

$$
\begin{aligned}
E\big(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) &= \lim_{n\to\infty} E\big(\|v_{G,Z_n}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big)\\
&= \lim_{n\to\infty} E\big(\|v_{(I-\Delta)^{\frac{\alpha}{2}}G,Z_n}(T)\|^2_{L^2(\mathbb{R}^d)}\big)\\
&= \lim_{n\to\infty} I^\alpha_{G,Z_n}\\
&= I^\alpha_{G,Z}.
\end{aligned}
$$

Step 3. Let us now put ourselves under the assumptions of the theorem. Let $(\psi_n, n \geq 1)$ be an approximation of the identity such that $|\mathcal{F}\psi_n(\xi)| \leq 1$, for all $\xi \in \mathbb{R}^d$. Set $G_n(s) = G(s) * \psi_n$, $G_n^\alpha = (I-\Delta)^{\frac{\alpha}{2}}G_n$. We now check that G_n and G_n^α satisfy **(G1')** and the assumption (iii) of the theorem.

It is clear that $G_n(s) \in \mathcal{C}^\infty(\mathbb{R}^d)$. Moreover, condition (ii) yields

$$\sup_{0\leq s\leq T}\sup_{x\in\mathbb{R}^d}|G_n(s,x)|) < \infty.$$

Since $\mathcal{F}G(s)$ is a function, $\mathcal{F}G_n(s)$ is also a function and (i) implies

$$\sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}|\mathcal{F}G_n(s)(\xi)| \leq \sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}|\mathcal{F}G(s)(\xi)| < \infty.$$

Notice that $G_n^\alpha(s) = G(s) * (I-\Delta)^{\frac{\alpha}{2}}\psi_n$. Since $(I-\Delta)^{\frac{\alpha}{2}}\psi_n \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of C^∞ test functions with rapid decrease, we have $G_n^\alpha \in \mathcal{C}^\infty(\mathbb{R}^d)$ (see for instance [8], Proposition 32.1.1).

The condition $\sup_{0\leq s\leq T}\sup_{x\in\mathbb{R}^d}|G_n^\alpha(s,x)| < \infty$ is a consequence of assumption (ii). Since $\mathcal{F}G(s)$ is a function, so is $\mathcal{F}G_n^\alpha(s)$. Moreover, condition (i) yields

$$\begin{aligned}
\sup_{n\geq 1}\sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}|\mathcal{F}G_n^\alpha(s)(\xi)| &= \sup_{n\geq 1}\sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}|\mathcal{F}G(s)(\xi)|\,|(1+|\xi|^2)^{\frac{\alpha}{2}}\mathcal{F}\psi_n(\xi)| \\
&\leq \sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}|(1+|\xi|^2)^{\frac{\alpha}{2}}|\mathcal{F}G(s)(\xi)| \\
&< \infty.
\end{aligned}$$

Consider the sequence of stochastic integrals $(v_{G_n,Z}(T), n\geq 1)$. Theorem 6 in [5] shows that $v_{G,Z}(T)$ is well-defined as an $L^2(\Omega; L^2(\mathbb{R}^d))$-valued random variable and

$$\begin{aligned}
&E(\|v_{G_n,Z}(T) - v_{G,Z}(T)\|^2_{L^2(\mathbb{R}^d)}) = I_{G_n-G,Z} \\
&= \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s)(\xi)|^2) \int_{\mathbb{R}^d}\mu(d\eta)|\mathcal{F}G(s)(\xi-\eta)|^2|\psi_n(\xi)-1|^2.
\end{aligned}$$

By dominated convergence, this expression tends to zero as n tends to infinity.

We want to prove that $(v_{G_n,Z}(T), n\geq 1)$ is a Cauchy sequence in $L^2(\Omega; H_2^\alpha(\mathbb{R}^d))$. Indeed, by the results stated in Step 2, we obtain

$$\begin{aligned}
&E(\|v_{G_n-G_m,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}) = I^\alpha_{G_n-G_m,Z} \\
&= \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s)(\xi))|^2) \\
&\times \int_{\mathbb{R}^d}\mu(d\eta)(1+|\xi-\eta|^2)^\alpha|\mathcal{F}G(s)(\xi)|\,|(\mathcal{F}\psi_n - \mathcal{F}\psi_m)(\xi)|^2.
\end{aligned}$$

This last expression tends to zero as n, m tend to infinity, by dominated convergence and assumption (iii). We have therefore proved that

$$\lim_{n\to\infty} E(\|v_{G_n,Z}(T) - v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}) = 0.$$

By the results of Step 2, we obtain

$$E\big(\|v_{G,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) = \lim_{n\to\infty} E\big(\|v_{G_n,Z}(T)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) = I^\alpha_{G,Z}.$$

This finishes the proof of the theorem. □

Example 1 *Consider the differential operator $\mathcal{L} = \partial^2_{tt} + (-\Delta)^{(k)}$, $k \in]0,\infty[$, and denote by G the fundamental solution of $\mathcal{L}u = 0$. It is easy to check that*

$$\mathcal{F}G(t)(\xi) = \frac{\sin(t|\xi|^k)}{|\xi|^k}. \tag{16}$$

Fix $\alpha \in [0,k[$ and assume that

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{k-\alpha}} < \infty. \tag{17}$$

Then the assumptions of Theorem 1 are satisfied.

Indeed, $\mathcal{F}G(s) \in \mathcal{C}^\infty(\mathbb{R}^d)$. Moreover,

$$\begin{aligned}\sup_{0\le s\le T}\sup_{\xi\in\mathbb{R}^d} &\Big((1+|\xi|^2)^{\frac{\alpha}{2}} \frac{\sin(t|\xi|^k)}{|\xi|^k}\Big)\\ &\le T \sup_{|\xi|\le 1}(1+|\xi|^2)^{\frac{\alpha}{2}} + \sup_{|\xi|\ge 1}(1+|\xi|^2)^{\frac{\alpha-k}{2}}.\end{aligned}$$

Since $\alpha < k$, this last expression is finite and thus condition (i) is satisfied.

Let $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then,

$$\begin{aligned}\sup_{0\le s\le T}\sup_{x\in\mathbb{R}^d} |(G(s)*\psi)(x)| &\le \sup_{0\le s\le T} \|\mathcal{F}(G(s)*\psi)\|_{L^1(\mathbb{R}^d)}\\ &\le T\|\mathcal{F}\psi\|_{L^1(\mathbb{R}^d)} < \infty,\end{aligned}$$

proving (ii).

Finally, we prove (iii). For any $k \in]0,\infty[$, it is easy to check that

$$\sup_{0\le s\le T} |\mathcal{F}G(s)(\xi)|^2 \le \frac{2^k(1+T^2)}{(1+|\xi|^2)^k}. \tag{18}$$

Therefore,

$$\sup_{0\le s\le T}\int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}G(s)(\xi-\eta)|^2 \le C\int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\xi-\eta|^2)^{k-\alpha}}, \tag{19}$$

where C is a positive constant depending on T and k.

Set $\gamma = k-\alpha$. We will show that

$$\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\xi-\eta|^2)^\gamma} \le \int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\eta|^2)^\gamma}. \tag{20}$$

Combining this property with assumption (17) yields (iii). Note for future reference that

$$\sup_{0\leq s\leq T}\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(d\eta)(1+|\xi-\eta|^2)^\alpha|\mathcal{F}G(s)(\xi-\eta)|^2<\infty. \tag{21}$$

In order to prove (20), set $\tau_y(x)=x+y$. Following an argument that appears in [10], observe that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\mu(d\eta)\,\frac{e^{-2\pi^2t|\eta|^2}}{(1+|\xi-\eta|^2)^\gamma}=\int_{\mathbb{R}^d}\Gamma(dx)\mathcal{F}^{-1}\big(e^{-2\pi^2t|\cdot|^2}\tau_{-\xi}(1+|\cdot|^2)^{-\gamma}\big)(x)\\
&=\int_{\mathbb{R}^d}\Gamma(dx)\big(\mathcal{F}^{-1}(e^{-2\pi^2t|\cdot|^2})*\mathcal{F}^{-1}(\tau_{-\xi}(1+|\cdot|^2)^{-\gamma})\big)(x)\\
&=\int_{\mathbb{R}^d}\Gamma(dx)(p_t*e_\xi G_{d,\gamma}(x),
\end{aligned}$$

where $p_t=\mathcal{F}^{-1}(e^{-2\pi^2t|\cdot|^2})$ is the Gaussian density with mean 0 and variance t, $e_\xi(x)=e^{2\pi i\langle x,\xi\rangle}$ and $G_{d,\gamma}(x)=\mathcal{F}^{-1}(1+|\cdot|^2)^{-\gamma}(x)$.

Since both p_t and $G_{d,\gamma}$ are positive functions,

$$|(p_t*e_\xi G_{d,\gamma}(x)|\leq\int_{\mathbb{R}^d}p_t(y)G_{d,\gamma}(x-y)dy.$$

Using monotone convergence and the fact that p_t is a probability density function on $\mathbb{R}^d$, we obtain

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{\mu(d\eta)}{(1+|\xi-\eta|^2)^\gamma}&=\sup_{\xi\in\mathbb{R}^d}\lim_{t\to0}\int_{\mathbb{R}^d}\mu(d\eta)\frac{e^{-2\pi^2t|\eta|^2}}{(1+|\xi-\eta|^2)^\gamma}\\
&\leq\lim_{t\to0}\int_{\mathbb{R}^d}\Gamma(dx)\int_{\mathbb{R}^d}G_{d,\gamma}(x-y)p_t(y)dy\\
&\leq\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(dx)G_{d,\gamma}(x-y).
\end{aligned}$$

However,

$$\sup_{y\in\mathbb{R}^d}\int_{\mathbb{R}^d}\Gamma(dx)G_{d,\gamma}(x-y)=\int_{\mathbb{R}^d}\frac{\mu(d\xi)}{(1+|\xi|^2)^\gamma}, \tag{22}$$

(see for instance [17], Lemma 9.8). This proves (20).

Example 2 *Let $\mathcal{L}$ and G be as in Example 1. Assume that $\Gamma(dx)=|x|^{-\beta}$, with $\beta\in]0,d[$. Then $\mu(d\xi)=C|\xi|^{-d+\beta}$ (see for instance [7]). Elementary calculations show that (17) holds provided that $\beta<2k$ and $\alpha\in[0,k-\frac{\beta}{2}[$.*

Fix $q\in]1,\infty[$ and $s\in]0,\infty[$. It is well known that the fractional Sobolev space $H^s_q(\mathbb{R}^d)$, is imbedded in the space $\mathcal{C}^\gamma(\mathbb{R}^d)$ of γ-Hölder continuous functions with $\gamma\leq s-\frac{d}{q}$, whenever $s-\frac{d}{q}>0$. Moreover, if $1<q<d$, $d>sq$, then $H^s_q(\mathbb{R}^d)$ is imbedded in $L^p(\mathbb{R}^d)$, for $q<p<\frac{dq}{d-sq}$ (see [19]). This yields the following.

Corollary 1 *1. Suppose that the assumptions of Theorem 1 are satisfied for some $\alpha \in]\frac{d}{2}, \infty[$. Then almost surely, the stochastic integral $v_{G,Z}(T)$ belongs to $\mathcal{C}^\gamma(\mathbb{R}^d)$, for any $\gamma \in]0, \alpha - \frac{d}{2}[$.*

2. If the hypotheses of Theorem 1 are satisfied with some $\alpha \in [0, \frac{d}{2}[$, then $E(\|v_{G,Z}(T)\|^2_{L^p(\mathbb{R}^d)}) < \infty$, for any $p \in]2, \frac{2d}{d-2\alpha}[$.

Remark 1 (a) *Let G be as in Example 1. Assume that condition (17) holds for some $\alpha \in]\frac{d}{2}, k[$. Then the conclusion of part 1 of Corollary 1 holds. This applies for instance to the wave equation in dimension $d = 1$.*

If condition (17) holds for some $\alpha \in [0, \frac{d}{2} \wedge k[$, then the conclusion of part 2 of Corollary 1 holds.

(b) *Let G be as in part (a) and $\Gamma(dx)$ be as in Example 2. Suppose that $\frac{d}{2} < k - \frac{\beta}{2}$. Then a.s., $v_{G,Z}(T)$ belongs to $\mathcal{C}^\gamma(\mathbb{R}^d)$, for any $\gamma \in]0, k - \frac{\beta+d}{2}[$.*

3 Path properties in time of the stochastic integral

We are now interested in the behaviour in t of the sample paths of the process $u_{G,Z} = (u_{G,Z}(t),\ t \in [0,T])$, where

$$u_{G,Z}(t) = \int_0^t \int_{\mathbb{R}^d} G(t-s, \cdot - y) Z(s,y) M(ds, dy).$$

We notice that in Theorem 1, one can replace everywhere the finite time horizon T by an arbitrary $t \in [0,T]$ and $G(s)$ by $G(t-s)$; therefore, under the assumptions of this theorem, the process $u_{G,Z}$ takes its values in the Hilbert space $H_2^\alpha(\mathbb{R}^d)$.

Our aim is to prove Hölder continuity of the sample paths. We shall apply a version of Kolmogorov's continuity condition; hence we are led to estimate L^p-moments of stochastic integrals with values in a Hilbert space by means of an extension of Burkholder's inequality. We therefore need to identify the increasing process associated with the martingale $(v_{G,Z}(t),\ t \in [0,T])$ of Theorem 1. We devote the first part of this section to this problem; the second part deals with the study of Hölder continuity.

3.1 The increasing process

Following [12], we term the *Meyer process* or *first increasing process* of the $H_2^\alpha(\mathbb{R}^d)$-valued martingale $(v_{G,Z}(t),\ t \in [0,T])$ the unique real-valued, continuous, increasing process, denoted by $(\langle v_{G,Z}\rangle_t,\ t \in [0,T])$, such that $\|v_{G,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} - \langle v_{G,Z}\rangle_t$ is a real-valued martingale.

Proposition 2 *Assume that the hypotheses of Theorem 1 are satisfied. Then for any $t \in [0,T]$,*

$$\langle v_{G,Z}\rangle_t = \int_0^t ds \int_{\mathbb{R}^d} d\xi\, |\mathcal{F}Z(s)(\xi)|^2 \\ \times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(s)(\xi-\eta)\right|^2. \qquad (23)$$

Proof. Assume first that G and Z satisfy the assumptions **(G1')**, **(G2)** and **(G3)** with T replaced by t, that is, $I^{0,t}_{G,Z} < \infty$, for any $t \in [0,T]$, where

$$I^{0,t}_{G,Z} = \int_0^t ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s)(\xi)|^2) \int_{\mathbb{R}^d} \mu(d\eta) |\mathcal{F}G(s)(\xi-\eta)|^2.$$

Suppose also that $G^\alpha(s) = (I-\Delta)^{\frac{\alpha}{2}} G(s)$ satisfies **(G1')** and $I^{\alpha,t}_{G,Z} < \infty$, where

$$I^{\alpha,t}_{G,Z} = \int_0^t ds \int_{\mathbb{R}^d} d\xi\, E(|\mathcal{F}Z(s)(\xi)|^2) \\ \times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}G(t-s)(\xi-\eta)|^2.$$

Then, following Lemma 1 in [5], for any $t \in [0,T]$ and $x \in \mathbb{R}^d$, the stochastic integral

$$v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(t,x) = \int_0^t \int_{\mathbb{R}^d} (I-\Delta)^{\frac{\alpha}{2}} G(s,x-y)(x) Z(s,y) M(ds,dy)$$

is well-defined as a Walsh stochastic integral. Its increasing process is given by

$$\langle v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}\rangle_t = \int_0^t ds \int_{\mathbb{R}^d} \mu(d\eta)\, |\mathcal{F}\big((I-\Delta)^{\frac{\alpha}{2}} G(s,x-\cdot) Z(s,\cdot)\big)(\eta)|^2,$$

(use Theorem 2.5 in [22] and elementary properties of convolution and the Fourier transform).

In particular, the process

$$|v_{(I-\Delta)^{\frac{\alpha}{2}} G,Z}(t,x)|^2 - \int_0^t ds \int_{\mathbb{R}^d} \mu(d\eta) \left|\mathcal{F}\big((I-\Delta)^{\frac{\alpha}{2}} G(s,x-\cdot) Z(s,\cdot)\big)(\eta)\right|^2, \qquad (24)$$

$t \in [0,T]$, is a real-valued martingale.

The properties of the Fourier transform yield

$$\mathcal{F}\big((I-\Delta)^{\frac{\alpha}{2}} G(s,x-\cdot) Z(s,\cdot)\big)(\eta) \\ = \Big(\mathcal{F}\big((I-\Delta)^{\frac{\alpha}{2}} G(s,x-\cdot)\big) * \mathcal{F}Z(s,\cdot)\Big)(\eta) \\ = \mathcal{F}\Big((1+|\xi'|^2)^{\frac{\alpha}{2}} \mathcal{F}G(s,\cdot)(\eta-\xi') \mathcal{F}Z(s,\cdot)(\xi')\Big)(x).$$

Then, by Plancherel's theorem,

$$\int_{\mathbb{R}^d} dx \int_0^t ds \int_{\mathbb{R}^d} \mu(d\eta) \left|\mathcal{F}((I-\Delta)^{\frac{\alpha}{2}} G(s, x-\cdot) Z(s,\cdot))(\eta)\right|^2$$
$$= \int_0^t ds \int_{\mathbb{R}^d} d\xi \, |\mathcal{F}Z(s)(\xi)|^2 \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(s)(\xi-\eta)\right|^2. \quad (25)$$

Following Step 1 in the proof of Theorem 1,

$$\|v_{G,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} = \|v_{(I-\Delta)^{\frac{\alpha}{2}} G, Z}(t)\|^2_{L^2(\mathbb{R}^d)},$$

for any $t \in [0,T]$.

Integrating over $x \in \mathbb{R}^d$ the expression in (24) and using (25), we find that the process

$$\|v_{G,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} - \int_0^t ds \int_{\mathbb{R}^d} d\xi \, |\mathcal{F}Z(s)(\xi)|^2$$
$$\times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(t-s)(\xi-\eta)\right|^2$$

is a real-valued martingale. This proves (23) under the particular set of assumptions stated at the beginning of the proof.

Assume next the setting of Step 2 in the proof of Theorem 1. That is, $G(s)$ and $G^\alpha(s)$ satisfy **(G1')** and condition (iii) holds. There exists a sequence of processes $(Z_n, n \geq 1)$ satisfying **(G2)** and $I^{\alpha,t}_{G,Z} < \infty$, such that for any $t \in [0,T]$,

$$\lim_{n\to\infty} E(\|v_{G,Z-Z_n}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)}) = \lim_{n\to\infty} I^{\alpha,t}_{G,Z_n-Z} = 0. \quad (26)$$

By the previous step,

$$M^n_t := \|v_{G,Z_n}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} - \int_0^t ds \int_{\mathbb{R}^d} d\xi \, |\mathcal{F}Z_n(s)(\xi)|^2$$
$$\times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(t-s)(\xi-\eta)\right|^2,$$

$t \in [0,T]$, is a real-valued martingale. Set

$$M_t = \|v_{G,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} - \int_0^t ds \int_{\mathbb{R}^d} d\xi \, |\mathcal{F}Z(s)(\xi)|^2$$
$$\times \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \left|\mathcal{F}G(t-s)(\xi-\eta)\right|^2.$$

From (26), it follows that $L^1(\Omega)$-$\lim_{n\to\infty} M^n_t = M_t$. This shows that $(M_t, t \in [0,T])$ is a martingale and proves (23) in the setting of Step 2.

Finally, we consider the situation given by the hypotheses of the theorem. From Step 3 in the proof of Theorem 1, it follows that for any $t \in [0,T]$,

$$\lim_{n\to\infty} E(\|v_{G-G_n,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)}) = \lim_{n\to\infty} I^{\alpha,t}_{G-G_n,Z} = 0,$$

where $G_n(s) = G(s) * \psi_n$ and $(\psi_n, n \geq 1)$ is an approximation of the identity. The sequence $(G_n(s),\ n \geq 1)$ satisfies the conditions of the previous step. Therefore, we can conclude using a limiting procedure, in a manner analoguous to the previous step. This completes the proof of the Proposition. □

Proposition 3 *Assume the hypotheses of Theorem 1. Fix $q \in [1, \infty[$. Then there is $C > 0$ such that for all $t > 0$,*

$$E(\|v_{G,Z}(t)\|^{2q}_{H_2^\alpha(\mathbb{R}^d)}) \leq C\, t^{q-1} \int_0^t ds\, E\left(\|Z(s)\|^{2q}_{L^2(\mathbb{R}^d)}\right) \times \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)(1 + |\xi - \eta|^2)^\alpha |\mathcal{F}G(s)(\xi - \eta)|^2\right)^q.$$

Proof. Using the Hilbert space version of Burkholder's inequality ([12], p. 212), Proposition 2, Hölder's inequality and Plancherel's identity, we obtain

$$\begin{aligned}
E(\|v_{G,Z}(t)\|^{2q}_{H_2^\alpha(\mathbb{R}^d)}) &\leq CE\Big(\Big(\int_0^t ds \int_{\mathbb{R}^d} d\xi\, |\mathcal{F}Z(s)(\xi)|^2 \int_{\mathbb{R}^d} \mu(d\eta)(1 + |\xi - \eta|^2)^\alpha \\
&\qquad \times |\mathcal{F}G(s)(\xi - \eta)|^2\Big)^q\Big) \\
&\leq Ct^{q-1} \int_0^t ds\, E\Big(\Big(\int_{\mathbb{R}^d} d\xi\, |\mathcal{F}Z(s)(\xi)|^2 \\
&\qquad \times \int_{\mathbb{R}^d} \mu(d\eta)(1 + |\xi - \eta|^2)^\alpha |\mathcal{F}G(s)(\xi - \eta)|^2\Big)^q\Big) \\
&\leq Ct^{q-1} \int_0^t ds\, E\left(\|Z(s)\|^{2q}_{L^2(\mathbb{R}^d)}\right) \\
&\qquad \times \left(\sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)(1 + |\xi - \eta|^2)^\alpha |\mathcal{F}G(s)(\xi - \eta)|^2\right)^q.
\end{aligned}$$

This proves the proposition. □

3.2 Hölder continuity in time

In this section, we consider the distribution-valued function $G(s)$ of Example 1. Our goal is to give sufficient conditions ensuring a.s.-Hölder continuity of the sample paths of the process $(u_{G,Z}(t), t \in [0, T])$ defined in (4).

We first study the case of a general covariance measure Γ. In a second part, we consider the particular case $\Gamma(dx) = |x|^{-\beta}$, $\beta \in]0, d[$. The radial structure of this measure makes it possible to obtain a higher order of Hölder continuity. Indeed, we prove that the result obtained in this situation is optimal.

Theorem 2 *Let $\mathcal{L} = \partial_{tt}^2 + (-\Delta)^k$, $k \in]0, \infty[$, and let G be the fundamental solution of $\mathcal{L}u = 0$. Fix $\alpha \in [0, k[$ and assume that there exists $\eta \in]\frac{\alpha}{k}, 1[$ such that*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{k\eta-\alpha}} < \infty. \tag{27}$$

Fix $q \in [2, \infty[$ and assume that $\sup_{0\le s\le T} E(\|Z(s)\|_{L^2(\mathbb{R}^d)}^q) < \infty$. Then the $H_2^\alpha(\mathbb{R}^d)$-valued stochastic integral process $(u_{G,Z}(t),\ 0 \le t \le T)$ satisfies

$$E(\|u_{G,Z}(t_2) - u_{G,Z}(t_1)\|_{H_2^\alpha(\mathbb{R}^d)}^q) \le C(t_2 - t_1)^{q\left(\frac{1}{2}\wedge(1-\eta)\right)}, \tag{28}$$

for any $0 \le t_1 \le t_2 \le T$. Consequently, $(u_{G,Z}(t), 0 \le t \le T)$ is γ-Hölder continuous, for each $\gamma \in\ ]0, (\frac{1}{2} \wedge (1-\eta)) - \frac{1}{q}[$.

Proof. Fix $0 \le t_1 \le t_2 \le T$ and set $q = 2p$. Then

$$E(\|u_{G,Z}(t_2) - u_{G,Z}(t_1)\|_{H_2^\alpha(\mathbb{R}^d)}^{2p}) \le C(T_1(t_1,t_2) + T_2(t_1,t_2)),$$

where

$$\begin{aligned} T_1(t_1,t_2) &= E(\| \int_{t_1}^{t_2} G(t_2 - s, \cdot - y) Z(s,y) M(ds,dy)\|_{H_2^\alpha(\mathbb{R}^d)}^{2p}), \\ T_2(t_1,t_2) &= E(\| \int_0^{t_1} (G(t_2 - s, \cdot - y) - G(t_1 - s, \cdot - y)) \\ &\qquad \times Z(s,y) M(ds,dy)\|_{H_2^\alpha(\mathbb{R}^d)}^{2p}). \end{aligned}$$

Arguing as in Proposition 3 and using (21), we obtain

$$\begin{aligned} T_1(t_1,t_2) &\le C(t_2-t_1)^{p-1} \int_{t_1}^{t_2} ds\, E\Big(\|Z(s)\|_{L^2(\mathbb{R}^d)}^{2p}\Big) \\ &\qquad \times \left(\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}G(t_2-s)(\xi-\eta)|^2\right)^p \\ &\le C(t_2-t_1)^p \sup_{0\le s\le T} E(\|Z(s)\|_{L^2(\mathbb{R}^d)}^{2p}) \\ &\qquad \times \sup_{0\le s\le T} \sup_{\xi\in\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}G(s)(\xi-\eta)|^2\right)^p \\ &\le C(t_2-t_1)^p. \end{aligned} \tag{29}$$

We now study the contribution of $T_2(t_1,t_2)$. Clearly,

$$\int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}(G(t_2-s)-G(t_1-s))(\xi-\eta)|^2 \le I_1(t_1,t_2)+I_2(t_1,t_2),$$

where

$$\begin{aligned} I_1(t_1,t_2) &= \int_{|\xi-\eta|\le 1} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}(G(t_2-s)-G(t_1-s))(\xi-\eta)|^2, \\ I_2(t_1,t_2) &= \int_{|\xi-\eta|> 1} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}(G(t_2-s)-G(t_1-s))(\xi-\eta)|^2. \end{aligned}$$

By (16), the mean-value theorem, the bound (20) and assumption (27),

$$\begin{aligned}
I_1(t_1,t_2) &\le (t_2-t_1)^2 \int_{|\xi-\eta|\le 1} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \\
&\le 2^k (t_2-t_1)^2 \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\xi-\eta|^2)^{k-\alpha}} \\
&\le C(t_2-t_1)^2 \int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\eta|^2)^{k-\alpha}} \\
&\le C(t_2-t_1)^2.
\end{aligned}$$

By the formula $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$,

$$\begin{aligned}
I_2(t_1,t_2) &\le \int_{|\xi-\eta|>1} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \frac{\left(\sin\left(\frac{1}{2}(t_2-t_1)|\xi-\eta|^k\right)\right)^{2(1-\eta)}}{|\xi-\eta|^{2k}} \\
&\le C(t_2-t_1)^{2(1-\eta)} \int_{\mathbb{R}^d} \frac{\mu(d\eta)}{(1+|\xi-\eta|^2)^{k\eta-\alpha}} \\
&\le C(t_2-t_1)^{2(1-\eta)}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\sup_{0\le s\le T}\sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha |\mathcal{F}(G(t_2-s)-G(t_1-s))(\xi-\eta)|^2 \\
&\le C(t_2-t_1)^{2(1-\eta)}. \qquad (30)
\end{aligned}$$

Using (30) and arguing as in the lines that led to (29), we see that

$$\begin{aligned}
T_2(t_1,t_2) &\le C \sup_{0\le s\le T} E(\|Z(s)\|^{2p}_{L^2(\mathbb{R}^d)}) \sup_{0\le s\le T}\sup_{\xi\in\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \mu(d\eta)(1+|\xi-\eta|^2)^\alpha \\
&\qquad \times |\mathcal{F}(G(t_2-s)-G(t_1-s))(\xi-\eta)|^2 \Big)^p \\
&\le C(t_2-t_1)^{2p(1-\eta)}. \qquad (31)
\end{aligned}$$

Finally, (28) is a consequence of (29) and (31).

The statement on Hölder continuity follows from Kolmogorov's continuity condition [15, Chap.I, §2]. □

The previous theorem, together with part 1 of Corollary 1, yields the following.

Corollary 2 *Suppose that the hypotheses of Theorem 2 are satisfied with some $\alpha \in]\frac{d}{2},\infty[$ and $\eta \in]\frac{\alpha}{k},1[$ Then there is a version of the process*

$$(u_{G,Z}(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d)$$

that belongs to $\mathcal{C}^{\gamma_1,\gamma_2}([0,T]\times\mathbb{R}^d)$, with $\gamma_1 \in]0,(\frac{1}{2}\wedge(1-\eta))-\frac{1}{q}[$ and $\gamma_2 \in]0,\alpha-\frac{d}{2}[$.

Consider now the particular case $\Gamma(dx) = |x|^{-\beta}$, $\beta \in]0, d[$. The results obtained in Theorem 2 can be improved as follows.

Theorem 3 *Let $\mathcal{L}$ and G be as in Theorem 2. Fix $\alpha \in [0, \infty[$, $k > \alpha$ and assume that $\Gamma(dx) = |x|^{-\beta}$ with $\beta \in]0, 2(k-\alpha)[$. Fix $q \in [2, \infty[$ and suppose that $\sup_{0\le s\le T} E(\|Z(s)\|^q_{L^2(\mathbb{R}^d)}) < \infty$. Then for any $0 \le t_1 \le t_2 \le 1$,*

$$E(\|u_{G,Z}(t_2) - u_{G,Z}(t_1)\|^q_{H_2^\alpha}) \le C(t_2 - t_1)^{q(1-\frac{\beta+2\alpha}{2k})}. \tag{32}$$

Consequently, $(u_{G,Z}(t),\ 0 \le t \le T)$ is γ-Hölder continuous for any $\gamma \in]0, (1 - \frac{\beta+2\alpha}{2k}) - \frac{1}{q}[$.

Remark 1 *If $\beta + 2\alpha < k$, then we obtain a stronger conclusion than in Theorem 2.*

Proof. As in the proof of Theorem 2, set $q = 2p$ and

$$T_1(t_1, t_2) = E(\|\int_{t_1}^{t_2} G(t_2 - s, \cdot - y)Z(s, y)M(ds, dy)\|^{2p}_{H_2^\alpha(\mathbb{R}^d)}),$$
$$T_2(t_1, t_2) = E(\|\int_0^{t_1} (G(t_2 - s, \cdot - y) - G(t_1 - s, \cdot - y))Z(s, y)M(ds, dy)\|^{2p}_{H_2^\alpha(\mathbb{R}^d)}).$$

Let

$$T_{11}(t_1, t_2) = \int_0^{t_2-t_1} ds \sup_{\xi\in\mathbb{R}^d} (\int_{\mathbb{R}^d} \mu(d\eta)(1 + |\xi - \eta|^2)^\alpha |\mathcal{F}G(s)(\xi - \eta)|^2)^p.$$

Then, proceeding as in the steps that led to (29), we find that

$$T_1(t_1, t_2) \le C(t_2 - t_1)^{p-1} \sup_{0\le s\le T} E(\|Z(s)\|^{2p}_{L^2(\mathbb{R}^d)})T_{11}(t_1, t_2).$$

Introducing the new variables $(\tilde\xi, \tilde\eta) = s^{\frac{1}{k}}(\xi, \eta)$ and substituting $|\eta|^{-d+\beta}$ for $\mu(d\eta)$ and formula (16) for $\mathcal{F}G(s)$ yields

$$T_{11}(t_1, t_2) = \int_0^{t_2-t_1} ds\, s^{p(2-\frac{\beta+2\alpha}{k})}$$
$$\times \sup_{\tilde\xi\in\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \frac{d\tilde\eta}{|\tilde\eta|^{d-\beta}} (s^{\frac{2}{k}} + |\tilde\xi - \tilde\eta|^2)^\alpha \frac{\sin^2(|\tilde\xi - \tilde\eta|^k)}{|\tilde\xi - \tilde\eta|^{2k}}\Big)^p.$$

Taking into account (19), (20) for $\mu(d\eta) = |\eta|^{-d+\beta}d\eta$ and the remark made in Example 2, we obtain

$$T_{11}(t_1, t_2) \le C(t_2 - t_1)^{p(2-\frac{\beta+2\alpha}{k})+1}.$$

Consequently,

$$T_1(t_1,t_2) \le C(t_2-t_1)^{p(3-\frac{\beta+2\alpha}{k})}. \tag{33}$$

For the analysis of the term $T_2(t_1,t_2)$, we also follow the same scheme as in the proof of Theorem 2 but we improve the upper bound on $I_2(t_1,t_2)$, as follows. Set $h = t_2 - t_1$ and consider the change of variables $(\tilde{\xi},\tilde{\eta}) = (\frac{h}{2})^{\frac{1}{k}}(\xi,\eta)$. Then,

$$\begin{aligned} I_2(t_1,t_2) &\le \int_{|\xi-\eta|>1} \frac{d\eta}{|\eta|^{d-\beta}}(1+|\xi-\eta|^2)^\alpha \frac{\sin^2\left(\frac{1}{2}(t_2-t_1)|\xi-\eta|^k\right)}{|\xi-\eta|^{2k}} \\ &\le Ch^{2-\frac{\beta+2\alpha}{k}} \int_{|\tilde{\xi}-\tilde{\eta}|>(\frac{h}{2})^{\frac{1}{k}}} \frac{d\tilde{\eta}}{|\tilde{\eta}|^{d-\beta}}(1+|\tilde{\xi}-\tilde{\eta}|^2)^\alpha \frac{\sin^2(|\tilde{\xi}-\tilde{\eta}|^k)}{|\tilde{\xi}-\tilde{\eta}|^{2k}}, \end{aligned}$$

where C is a constant depending on k. Therefore,

$$T_2(t_1,t_2) \le C(t_2-t_1)^{p(2-\frac{\beta+2\alpha}{k})}. \tag{34}$$

The estimates (33) and (34) imply (32). The proof of the theorem is complete. $\square$

We finish this section by showing that Theorem 3 provides an optimal result. We do this by studying the case where Z is the smooth deterministic function $Z(s,x) = e^{-|x|^2/2}$, with no dependence on s. For this Z, we shall write $u_G(t)$ instead of $u_{G,Z}(t)$.

Theorem 4 *Let $\mathcal{L}$, G and Γ be as in Theorem 3. Fix $t_0 \in (0,1]$ and assume $\beta \in]0, 2(k-\alpha)[$. Then there exists a constant $C > 0$ such that for any t_1, t_2 satisfying $t_0 \le t_1 \le t_2 \le 1$,*

$$E\big(\|u_G(t_2) - u_G(t_1)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) \ge C|t_2-t_1|^{2-\frac{\beta+2\alpha}{k}}. \tag{35}$$

Consequently, a.s. the mapping $t \mapsto u_G(t)$ is not *γ-Hölder continuous for $\gamma > 1-(\beta+2\alpha)/(2k)$, while it is γ-Hölder continuous for $\gamma < 1-(\beta+2\alpha)/(2k)$.*

Proof. Let $p(\xi)$ denote the standard Gaussian density function. Using the isometry property (7), we obtain

$$E\big(\|u_G(t_2) - u_G(t_1)\|^2_{H_2^\alpha(\mathbb{R}^d)}\big) \ge S(t_1,t_2),$$

where

$$\begin{aligned} S(t_1,t_2) = \int_0^{t_1} ds \int_{\mathbb{R}^d} d\xi\, p(\xi)^2 \int_{\mathbb{R}^d} \mu(d\eta)\big(1+|\xi-\eta|^2\big)^\alpha \\ \times \big|\mathcal{F}G(t_2-s)(\xi-\eta) - \mathcal{F}G(t_1-s)(\xi-\eta)\big|^2. \end{aligned} \tag{36}$$

Set $h = (t_2 - t_1)/2$. By the formula $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$, Fubini's theorem and integrating with respect to the time variable s, we obtain

$$S(t_1,t_2) = 4\int_{\mathbb{R}^d} d\xi\, p(\xi)^2 \int_{\mathbb{R}^d} \frac{d\eta}{|\eta|^{d-\beta}} (1+|\xi-\eta|^2)^\alpha \frac{\sin^2(h|\xi-\eta|^k)}{|\xi-\eta|^{2k}} \times \left(\frac{t_1}{2} - \frac{\sin((t_2-t_1)|\xi-\eta|^k)}{4|\xi-\eta|^k} + \frac{\sin((t_1+t_2)|\xi-\eta|^k)}{4|\xi-\eta|^k} \right).$$

Notice that for $|\xi-\eta|^k > 2/t_1$ and, in particular, for $|\eta| > 2(2/t_1)^{1/k}$ and $|\xi| < (2/t_1)^{1/k}$, the factor in parentheses is bounded below by $t_1/4$. Therefore,

$$S(t_1,t_2) \geq t_0 \int_{|\xi|<(2/t_1)^{1/k}} d\xi\, p(\xi)^2 \int_{|\eta|>2(2/t_1)^{1/k}} \frac{d\eta}{|\eta|^{d-\beta}} \times (1+|\xi-\eta|^2)^\alpha \frac{\sin^2(h|\xi-\eta|^k)}{|\xi-\eta|^{2k}}.$$

Let $a = (2/t_1)^{1/k}$. Note that

$$\{(\xi,\eta) : |\xi| < a,\ |\eta| > 2a\} \supset \{(\xi,\eta) : |\xi| < a,\ |\xi-\eta| > 3a\}$$

and that $1/|\eta| > 1/(2|\xi-\eta|)$ for (ξ,η) in these sets. With this inequality and this smaller domain of integration, we use the change of variables $\tilde\eta = h^{\frac{1}{k}}(\xi-\eta)$ (ξ fixed) to see that

$$\begin{aligned} S(t_1,t_2) &\geq \frac{t_0}{2} h^{2-\frac{\beta+2\alpha}{k}} \int_{|\xi|<a} d\xi\, p(\xi)^2 \int_{|\tilde\eta|^k > 3^k a^k} \frac{d\tilde\eta}{|\tilde\eta|^{d-\beta+2k}} (h^{\frac{2}{k}} + |\tilde\eta|^2)^\alpha \sin^2(|\tilde\eta|) \\ &\geq \frac{t_0}{2} h^{2-\frac{\beta+2\alpha}{k}} \int_{|\xi|<2^{1/k}} d\xi\, p(\xi)^2 \int_{|\tilde\eta|^k > 3^k 2/t_0} \frac{d\tilde\eta}{|\tilde\eta|^{d-\beta+2(k-\alpha)}} \sin^2(|\tilde\eta|). \end{aligned}$$

Notice that the last double integral is a positive finite constant. Hence, the inequality (35) is proved.

We now use the fact that u_G is a Gaussian stationary process together with classical results on Gaussian processes to translate the lower bound (35) into a statement concerning absence of Hölder continuity of the sample paths of $t \mapsto u_G(t)$. Fix $\gamma \in]1-(\beta+2\alpha)/(2k), 1]$ and assume by contradiction that for almost all ω, there is $C(\omega) < \infty$ such that for all $t_0 \leq t_1 < t_2 \leq 1$,

$$\begin{aligned} &\sup_{t_1<t_2} \sup_{\varphi \in H_2^{-\alpha}(\mathbb{R}^d),\, \varphi \not\equiv 0} \frac{\langle u_G(t_2) - u_G(t_1),\ \varphi \rangle}{(t_2-t_1)^\gamma \, \|\varphi\|_{H_2^{-\alpha}(\mathbb{R}^d)}} \\ &\qquad = \sup_{t_1<t_2} \frac{\|u_G(t_2) - u_G(t_1)\|_{H_2^{\alpha}(\mathbb{R}^d)}}{(t_2-t_1)^\gamma} \\ &\qquad < C(\omega). \end{aligned}$$

Then the real-valued Gaussian stochastic process

$$\left(\frac{\langle u_G(t_2) - u_G(t_1),\ \varphi \rangle}{(t_2 - t_1)^\gamma \, \|\varphi\|_{H_2^{-\alpha}(\mathbb{R}^d)}}, \quad t_1 < t_2,\ \varphi \in H_2^{-\alpha}(\mathbb{R}^d),\, \varphi \not\equiv 0 \right)$$

is finite a.s. By Theorem 3.2 of [1], it follows that

$$E\left(\sup_{t_1 < t_2} \sup_{\varphi \in H_2^{-\alpha}(\mathbb{R}^d),\, \varphi \not\equiv 0} \left(\frac{\langle u_G(t_2) - u_G(t_1),\ \varphi \rangle}{(t_2 - t_1)^\gamma \, \|\varphi\|_{H_2^{-\alpha}(\mathbb{R}^d)}} \right)^2 \right) < \infty.$$

Thus,

$$E\left(\sup_{t_1 < t_2} \frac{\|u_G(t_2) - u_G(t_1)\|^2_{H_2^\alpha(\mathbb{R}^d)}}{(t_2 - t_1)^{2\gamma}} \right) < \infty.$$

In particular, there would exist $K < \infty$ such that

$$E\left(\|u_G(t_2) - u_G(t_1)\|^2_{H_2^\alpha(\mathbb{R}^d)} \right) \leq K\, |t_2 - t_1|^{2\gamma}.$$

However, this would contradict (35) since $2\gamma > 2 - (\beta + 2\alpha)/k$. We conclude that $t \mapsto u_G(t)$ is *not* γ-Hölder continuous for $\gamma > 1 - (\beta + 2\alpha)/(2k)$.

On the other hand, for $\gamma < 1 - (\beta + 2\alpha)/(2k)$, the map $t \mapsto u_G(t)$ is γ-Hölder continuous by Theorem 3, since in this theorem, q can be taken arbitrarily large. □

4 Application to stochastic partial differential equations

This section is devoted to studying the properties of the sample paths of the solution of the spde

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2} + (-\Delta)^{(k)} \right) u(t,x) = \sigma\big(u(t,x)\big)\dot{F}(t,x) + b\big(u(t,x)\big), \\ &u(0,x) = v_0(x), \qquad \frac{\partial}{\partial t} u(0,x) = \tilde{v}_0(x). \end{aligned} \tag{37}$$

In this equation, $t \in [0,T]$ for some fixed $T > 0$, and $x \in \mathbb{R}^d$. We assume that $k \in \,]0,\infty[$ (k is not necessarily an integer), σ and b are Lipschitz continuous functions and moreover, that

$$|\sigma(z)| + |b(z)| \leq C|z|, \tag{38}$$

for some positive constant $C > 0$. Notice that the assumption (38) was also made in [5] and it is also standard in the study of the deterministic wave equation (see for instance [9, Chapter 6] or [20]).

Concerning the initial conditions, we assume for the moment that $v_0 \in L^2(\mathbb{R}^d)$, $\tilde{v}_0 \in H_2^{-k}(\mathbb{R}^d)$. Regarding the noise $\dot{F}$, we assume that its spectral measure satisfies (17).

By a solution of (37), we mean an $L^2(\mathbb{R}^d)$-valued stochastic process $\big(u(t),\ 0 \le t \le T\big)$ satisfying $\sup_{0\le t\le T} E\big(\|u(t)\|^2_{L^2(\mathbb{R}^d)}\big) < \infty$ and

$$\begin{aligned} u(t,\cdot) = {} & \frac{d}{dt} G(t) * v_0 + G(t) * \tilde{v}_0 \\ & + \int_0^t \int_{\mathbb{R}^d} G(t-s,\cdot - y)\sigma(u(s,y))M(ds,dy) \\ & + \int_0^t ds \int_{\mathbb{R}^d} dy\, G(t-s,\cdot - y) b(u(s,y)). \end{aligned} \tag{39}$$

Here, G is the fundamental solution of $\mathcal{L}f = 0$, where $\mathcal{L} = \big(\partial^2_{tt} + (-\Delta)^{(k)}\big)$, and the stochastic integral is of the type considered in the preceding sections (see also Section 2 in [5]).

The path integral is also well-defined. Indeed, let $Z = (Z(s),\ s \in [0,T])$ be a stochastic processes satisfying the conditions stated at the beginning of Section 2, and let $G : [0,T] \to \mathcal{S}'(\mathbb{R}^d)$ be such that for any $s \in [0,T]$, $\mathcal{F}G(s)$ is a function and

$$\int_0^T ds \sup_{\xi \in \mathbb{R}^d} |\mathcal{F}G(s)(\xi)|^2 < \infty. \tag{40}$$

Then for any $t \in [0,T]$ a.s.,

$$x \mapsto J_{G,Z}(t,x) := \int_0^t ds\, \big(G(s) * Z(s)\big)(x)$$

defines an $L^2(\mathbb{R}^d)$-valued function. Moreover,

$$\|J_{G,Z}(t)\|^2_{L^2(\mathbb{R}^d)} \le C \int_0^t ds\, \|Z(s)\|^2_{L^2(\mathbb{R}^d)} \sup_{\xi \in \mathbb{R}^d} |\mathcal{F}G(s)(\xi)|^2. \tag{41}$$

Assume the following condition, which is stronger than (40):

$$\int_0^T ds \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^\alpha |\mathcal{F}G(s)(\xi)|^2 < \infty, \tag{42}$$

for some $\alpha \in [0, \infty[$. Easy computations based on Fubini's theorem yield

$$\|J_{G,Z}(t)\|^2_{H_2^\alpha(\mathbb{R}^d)} = \|J_{(I-\Delta)^{\frac{\alpha}{2}}G,Z}(t)\|^2_{L^2(\mathbb{R}^d)} \qquad \text{a.s.}$$

Fix $q \in [2, \infty[$. Schwarz's inequality and Fubini's theorem yield a.s.:

$$\begin{aligned}
\|J_{(I-\Delta)^{\frac{\alpha}{2}}G,Z}(t)\|^q_{L^2(\mathbb{R}^d)} &\le C\left(\int_0^t ds\, \|(I-\Delta)^{\frac{\alpha}{2}}G(s) * Z(s)\|^2_{L^2(\mathbb{R}^d)}\right)^{\frac{q}{2}} \\
&= C\left(\int_0^t ds \int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^\alpha |\mathcal{F}G(s)(\xi)|^2 |\mathcal{F}Z(s)(\xi)|^2\right)^{\frac{q}{2}} \\
&\le C\left(\int_0^t ds\, \|Z(s)\|^2_{L^2(\mathbb{R}^d)} \sup_{\xi\in\mathbb{R}^d}\left((1+|\xi|^2)^\alpha |\mathcal{F}G(s)(\xi)|^2\right)\right)^{\frac{q}{2}} \\
&\le C\int_0^t ds\, \|Z(s)\|^q_{L^2(\mathbb{R}^d)} \sup_{\xi\in\mathbb{R}^d}\left((1+|\xi|^2)^{\frac{\alpha q}{2}} |\mathcal{F}G(s)(\xi)|^q\right).
\end{aligned}$$

Now let G be the fundamental solution of $\mathcal{L}f = 0$. Assume that $\alpha \in [0, k[$ and $\sup_{t\in[0,T]} E(\|Z(t)\|^q_{L^2(\mathbb{R}^d)}) < \infty$. Then

$$\sup_{\xi\in\mathbb{R}^d}(1+|\xi|^2)^{\frac{\alpha q}{2}}|\mathcal{F}G(s)(\xi)|^q \le C\sup_{\xi\in\mathbb{R}^d}(1+|\xi|^2)^{q(\alpha-k)/2} < \infty,$$

and therefore the above inequalities yield

$$\begin{aligned}
E(\|J_{(I-\Delta)^{\frac{\alpha}{2}}G,Z}(t)\|^q_{L^2(\mathbb{R}^d)}) &\le C\int_0^t ds \sup_{\xi\in\mathbb{R}^d}(1+|\xi|^2)^{\frac{\alpha q}{2}}|\mathcal{F}G(s)(\xi)|^q \\
&\qquad \times E(\|Z(s)\|^q_{L^2(\mathbb{R}^d)}) \\
&< \infty. \qquad (43)
\end{aligned}$$

Set

$$J_b(t) = \int_0^t ds \int_{\mathbb{R}^d} dy\, G(t-s, \cdot - y) b(u(s,y)) = \int_0^t ds\, \big(G(s) * b(u(s))\big).$$

Particularizing (43) to $\alpha = 0$, $q = 2$ and $Z(s,x) = b(u(t-s,x))$ yields

$$E\big(\|J_b(t)\|^2_{L^2(\mathbb{R}^d)}\big) \le C \sup_{0\le s\le T} E\big(\|u(s)\|^2_{L^2(\mathbb{R}^d)}\big)\int_0^T ds \sup_{\xi\in\mathbb{R}^d}|\mathcal{F}G(s)(\xi)|^2 < \infty.$$

A slight extension of Theorem 9 in [5] provides the existence of a unique solution of equation (39), in the sense given above. We observe that in [5], $k \in \mathbb{N}$ and $b = 0$.

By means of Burkholder's and Hölder's inequalities (as in the calculation that led to (29)), the inequality (43) with $\alpha = 0$ and a version of Gronwall's lemma (see [4, Lemma 15]), one can easily show that for any $q \in [2, \infty[$,

$$\sup_{0\le s\le T} E\big(\|u(s)\|^q_{L^2(\mathbb{R}^d)}\big) < \infty. \qquad (44)$$

In the next theorem, we analyze the existence of $H_2^\alpha(\mathbb{R}^d)$-valued solutions to (39).

Theorem 5 *Let σ, b be real-valued Lipschitz continuous functions satisfying (38). Fix $\alpha \in [0,k[$ and assume that $v_0 \in H_2^{\alpha}(\mathbb{R}^d)$, $\tilde{v}_0 \in H_2^{\alpha-k}(\mathbb{R}^d)$ and*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{k-\alpha}} < \infty.$$

Then for any $q \in [2,\infty[$, the solution of (39) satisfies

$$\sup_{0\le t\le T} E\big(\|u(t)\|^q_{H_2^{\alpha}(\mathbb{R}^d)}\big) < \infty. \tag{45}$$

Proof. Fix $q \in [2,\infty[$. We shall check that each term on the right hand side of (39) belongs to $L^q(\Omega; H_2^{\alpha}(\mathbb{R}^d))$, with norm uniformly bounded over $t \in [0,T]$.

Set $U_1(t) = \frac{d}{dt}G(t) * v_0$. Then

$$\begin{aligned}
\|U_1(t)\|_{H_2^{\alpha}(\mathbb{R}^d)} &= \big\|(1+|\cdot|^2)^{\frac{\alpha}{2}}\mathcal{F}(\frac{d}{dt}G(t)(\cdot))\mathcal{F}v_0(\cdot)\big\|_{L^2(\mathbb{R}^d)} \\
&= \big\|(1+|\cdot|^2)^{\frac{\alpha}{2}}\cos(t|\cdot|^k)\mathcal{F}v_0(\cdot)\big\|_{L^2(\mathbb{R}^d)} \\
&\le \|v_0\|_{H_2^{\alpha}(\mathbb{R}^d)}.
\end{aligned}$$

Similarly, define $U_2(t) = G(t) * \tilde{v}_0$. Then, by (18),

$$\begin{aligned}
\|U_2(t)\|^2_{H_2^{\alpha}(\mathbb{R}^d)} &= \big\|(1+|\cdot|^2)^{\frac{\alpha}{2}}\mathcal{F}G(t)(\cdot)\mathcal{F}\tilde{v}_0(\cdot)\big\|^2_{L^2(\mathbb{R}^d)} \\
&\le 2^k(1+T^2)\|\tilde{v}_0\|^2_{H_2^{\alpha-k}(\mathbb{R}^d)}.
\end{aligned}$$

Hence,

$$\sup_{0\le t\le T}\big(\|U_1(t)\|^2_{H_2^{\alpha}(\mathbb{R}^d)} + \|U_2(t)\|^2_{H_2^{\alpha}(\mathbb{R}^d)}\big) < \infty. \tag{46}$$

Let

$$U_3(t) = \int_0^t \int_{\mathbb{R}^d} G(t-s,\cdot-y)\sigma(u(s,y))M(ds,dy).$$

Using (38), we see as in (29) that

$$\begin{aligned}
E\big(\|U_3(t)\|^q_{H_2^{\alpha}(\mathbb{R}^d)}\big) &\le C \sup_{0\le s\le T} E\big(\|u(s)\|^q_{L^2(\mathbb{R}^d)}\big) \\
&\quad \times \sup_{0\le s\le T}\sup_{\xi\in\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\mu(d\eta)(1+|\xi-\eta|^2)^{\alpha}|\mathcal{F}G(s)(\xi-\eta)|^2\Big)^{\frac{q}{2}}.
\end{aligned}$$

By (21) and (44),

$$\sup_{0\le t\le T} E\big(\|U_3(t)\|^q_{H_2^{\alpha}(\mathbb{R}^d)}\big) < \infty. \tag{47}$$

Finally, set

$$U_4(t) = \int_0^t ds \int_{\mathbb{R}^d} dy\, G(t-s,\cdot-y)b(u(s,y)).$$

The estimate (43), (38) and (44) imply

$$\sup_{0\le t\le T} E\left(\|U_4(t)\|^q_{H_2^\alpha(\mathbb{R}^d)}\right) < \infty. \tag{48}$$

With (46)-(48), we finish the proof of the theorem. □

The next results concern the sample path properties of the solution of (39).

Theorem 6 *Let σ, b be Lipschitz functions satisfying (38). Fix $k \in]0,\infty[$, $\alpha \in [0,k[$ and assume that there exists $\eta \in]\frac{\alpha}{k}, 1[$ such that*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{(1+|\xi|^2)^{k\eta-\alpha}} < \infty. \tag{49}$$

Suppose also that $v_0 \in H_2^{k\delta+\alpha}(\mathbb{R}^d)$, for some $\delta \in]0,1]$ and $\tilde v_0 \in H_2^{-(k\gamma-\alpha)}(\mathbb{R}^d)$, for some $\gamma \in [0,1[$. Set $\theta_0 = \inf(\frac{1}{2}, 1-\eta, \delta, 1-\gamma)$. Then, for any $q \in [2,\infty[$ and $0 \le s \le t \le T$,

$$E\left(\|u(t)-u(s)\|^q_{H_2^\alpha(\mathbb{R}^d)}\right) \le C(t-s)^{q\theta},$$

with $\theta \in]0,\theta_0[$. Therefore, the sample paths of the $H_2^\alpha(\mathbb{R}^d)$-valued process $(u(t),\ t\in[0,T])$ solution of (39) are almost surely θ-Hölder continuous for any $\theta \in]0,\theta_0[$.

Proof. Fix $0 \le s \le t \le 1$. As in the proof of Theorem 5, let $U_1(t) = \frac{d}{dt}G(t) * v_0$. Using the formula $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$, we obtain

$$\begin{aligned}
\|U_1(t)-U_1(s)\|^2_{H_2^\alpha(\mathbb{R}^d)} &= \|(1+|\cdot|^2)^{\frac{\alpha}{2}}(\cos(t|\cdot|^k)-\cos(s|\cdot|^k))\mathcal{F}v_0(\cdot)\|^2_{L^2(\mathbb{R}^d)}\\
&\le 4\int_{\mathbb{R}^d} d\xi\,(1+|\xi|^2)^\alpha \left(\sin\frac{(t-s)|\xi|^k}{2}\right)^{2\delta} |\mathcal{F}v_0(\xi)|^2\\
&\le (t-s)^{2\delta}\|v_0\|^2_{H_2^{k\delta+\alpha}(\mathbb{R}^d)}\\
&\le C(t-s)^{2\delta}.
\end{aligned}\tag{50}$$

Consider now the term $U_2(t) = G(t) * \tilde v_0$. Applying the formula $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$ yields

$$\begin{aligned}
\|U_2(t)-U_2(s)\|^2_{H_2^\alpha(\mathbb{R}^d)} &\le \|(1+|\cdot|^2)^{\frac{\alpha}{2}}\left(\frac{\sin\frac{(t-s)|\cdot|^k}{2}}{|\cdot|^k}\right)\mathcal{F}\tilde v_0(\cdot)\|^2_{L^2(\mathbb{R}^d)}\\
&\le T_1 + T_2,
\end{aligned}$$

where

$$T_1 = (t-s)^2 \int_{\mathbb{R}^d \cap \{|\xi|<1\}} d\xi \, (1+|\xi|^2)^\alpha \frac{|\mathcal{F}\tilde{v}_0(\xi)|^2}{|\xi|^{2k}},$$

$$T_2 = \int_{\mathbb{R}^d \cap \{|\xi|\geq 1\}} d\xi \, (1+|\xi|^2)^\alpha |\mathcal{F}\tilde{v}_0(\xi)|^2 \frac{\left(\sin \frac{(t-s)|\xi|^k}{2}\right)^{2(1-\gamma)}}{|\xi|^{2k}}.$$

Therefore,

$$\begin{aligned} T_1 + T_2 &\leq C_1(t-s)^2 \int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|^2)^{k-\alpha}} |\mathcal{F}\tilde{v}_0(\xi)|^2 \\ &\quad + C_2(t-s)^{2(1-\gamma)} \int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|^2)^{k\gamma-\alpha}} |\mathcal{F}\tilde{v}_0(\xi)|^2 \\ &\leq C(t-s)^{2(1-\gamma)} \|\tilde{v}_0\|^2_{H_2^{-(k\gamma-\alpha)}(\mathbb{R}^d)}. \end{aligned}$$

Consequently, by the assumption on $\tilde{v}_0$,

$$\|U_2(t) - U_2(s)\|_{H_2^\alpha(\mathbb{R}^d)} \leq C(t-s)^{1-\gamma}. \tag{51}$$

Set

$$U_3(t) = \int_0^t \int_{\mathbb{R}^d} G(t-r, \cdot - y)\sigma(u(r,y))M(dr,dy).$$

For any $q \in [2, \infty[$, the following estimate holds:

$$E\big(\|U_3(t) - U_3(s)\|^q_{H_2^\alpha(\mathbb{R}^d)}\big) \leq C(t-s)^{q(\frac{1}{2}\wedge(1-\eta))}. \tag{52}$$

Indeed, set $Z(s,y) = \sigma(u(s,y))$. Properties (38) and (44) imply that

$$\sup_{0\leq s\leq T} E\big(\|\sigma(u(s))\|^q_{L^2(\mathbb{R}^d)}\big) < \infty.$$

Hence, (52) follows from the upper bound estimate (28).

Finally, set

$$U_4(t) = \int_0^t ds \int_{\mathbb{R}^d} dy \, G(t-s, \cdot - y) b(u(s,y)).$$

Clearly,

$$E\big(\|U_4(t) - U_4(s)\|^q_{H_2^\alpha(\mathbb{R}^d)}\big) \leq A(s,t) + B(s,t),$$

where

$$A(s,t) = E\big(\|\int_s^t dr \int_{\mathbb{R}^d} dy \, G(t-r, \cdot - y) b(u(r,y))\|^q_{H_2^\alpha(\mathbb{R}^d)}\big)$$

$$B(s,t) = E\big(\|\int_0^s dr \int_{\mathbb{R}^d} dy \, (G(t-r, \cdot - y) - G(s-r, \cdot - y)) b(u(r,y))\|^q_{H_2^\alpha(\mathbb{R}^d)}\big).$$

The Cauchy-Schwarz inequality, Hölder's inequality, Plancherel's identity, (38) and the fact that $\alpha < k$ yield

$$\begin{aligned}
A(s,t) &= E\Big(\Big(\int_{\mathbb{R}^d} dx\, \Big|\int_s^t dr \int_{\mathbb{R}^d} dy\, (I-\Delta)^{\frac{\alpha}{2}} G(t-r,x-y) b(u(r,y))\Big|^2\Big)^{\frac{q}{2}}\Big)\\
&\le (t-s)^{\frac{q}{2}} E\Big(\Big(\int_s^t dr \int_{\mathbb{R}^d} dx\, \Big|\int_{\mathbb{R}^d} dy\, (I-\Delta)^{\frac{\alpha}{2}} G(t-r,x-y) b(u(r,y))\Big|^2\Big)^{\frac{q}{2}}\Big)\\
&\le (t-s)^{q-1} \int_s^t dr\, E\Big(\Big(\int_{\mathbb{R}^d} dx\, \Big|\int_{\mathbb{R}^d} dy\, (I-\Delta)^{\frac{\alpha}{2}} G(t-r,x-y) b(u(r,y))\Big|^2\Big)^{\frac{q}{2}}\Big)\\
&= (t-s)^{q-1} \int_s^t dr\, E\Big(\Big(\int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^{\alpha} |\mathcal{F}G(t-r)(\xi)|^2 |\mathcal{F}b(u(r))(\xi)|^2\Big)^{\frac{q}{2}}\Big)\\
&\le C(t-s)^q.
\end{aligned}$$

Analogously, by the formula $\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$,

$$\begin{aligned}
B(s,t) &\le C\int_0^s dr E\Big(\Big(\int_{\mathbb{R}^d} dx\, \Big|\int_{\mathbb{R}^d} dy\, (I-\Delta)^{\frac{\alpha}{2}} (G(t-r,x-y) - G(s-r,x-y))\\
&\qquad \times b(u(r,y))\Big|^2\Big)^{\frac{q}{2}}\Big)\\
&\le C\int_0^s dr\, E\Big(\Big(\int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^{\alpha} \Big|\frac{\sin\frac{(t-s)|\xi|^k}{2}}{|\xi|^k}\Big|^2 |\mathcal{F}b(u(r))(\xi)|^2\Big)^{\frac{q}{2}}\Big)\\
&\le C\int_0^s dr\, E\Big(\Big(\int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^{\alpha} \frac{(\sin\frac{(t-s)|\xi|^k}{2})^{2(1-\eta)}}{|\xi|^{2k}}\, |\mathcal{F}b(u(r))(\xi)|^2\Big)^{\frac{q}{2}}\Big)\\
&\le (t-s)^{q(1-\eta)} \int_0^s dr\, E\Big(\Big(\int_{\mathbb{R}^d} d\xi\, (1+|\xi|^2)^{\alpha-k\eta}\, |\mathcal{F}b(u(r))(\xi)|^2\Big)^{\frac{q}{2}}\Big)\\
&\le C(t-s)^{q(1-\eta)},
\end{aligned}$$

because $\alpha - k\eta < 0$. Consequently,

$$E(\|U_4(t) - U_4(s)\|^q_{H_2^{\alpha}(\mathbb{R}^d)}) \le C(t-s)^{q(1-\eta)}. \tag{53}$$

The result then follows from (50)-(53). □

We finish this section with a refinement of the previous theorem in the particular case of a covariance measure Γ given by a Riesz kernel.

Theorem 7 *Fix $k \in]0,\infty[$ and $\alpha \in [0,k[$. Let σ, b, v_0, $\tilde{v}_0$, δ and γ be as in Theorem 6. We assume that $\Gamma(dx) = |x|^{-\beta}$, with $\beta \in]0, 2(k-\alpha)[$. Set $\theta_1 \in]0, \inf(1 - \frac{\beta+2\alpha}{2k}, \delta, 1-\gamma)[$. Then, for any $q \in [2,\infty[$, $0 \le s \le t \le T$,*

$$E(\|u(t) - u(s)\|^q_{H_2^{\alpha}(\mathbb{R}^d)}) \le C(t-s)^{q\theta}, \tag{54}$$

with $\theta \in]0, \theta_1[$. Therefore, the sample paths of the $H_2^{\alpha}(\mathbb{R}^d)$-valued process $(u(t),\ t \in [0,T])$ solution of (39) are almost surely θ-Hölder continuous for any

$$\theta \in \Big]0, \inf\Big(1 - \frac{\beta+2\alpha}{2k}, \delta, 1-\gamma\Big)\Big[.$$

Proof. We shall use the same notations as in the proof of Theorem 6. By Theorem 3 with $Z(s,y) = \sigma(u(s,y))$ (see (32)),

$$E(\|U_3(t) - U_3(s)\|^q_{H_2^\alpha(\mathbb{R}^d)}) \leq C(t_2 - t_1)^{q(1-\frac{\beta+2\alpha}{2k})}. \tag{55}$$

It is easy to check that for $\mu(d\xi) = |\xi|^{-d+\beta}$, the condition (49) holds in fact for any $\eta \in](2\alpha+\beta)/(2k), 1[$. Consequently, (53) yields

$$E(\|U_4(t) - U_4(s)\|^q_{H_2^\alpha(\mathbb{R}^d)}) \leq C(t_2 - t_1)^{q\theta_2}, \tag{56}$$

for any $\theta_2 \in]0, 1 - (2\alpha+\beta)/(2k)[$.

The upper bound estimate (54) is a consequence of (50), (51), (55), (56) and Hölder continuity of the $H_2^\alpha(\mathbb{R}^d)$-valued process $(u(t),\ t \in [0,T])$ follows from Kolmogorov's continuity condition. $\square$

Acknowledgement. The second named author thanks the *Institut de Mathématiques* of the *Ecole Polytechnique Fédérale de Lausanne* for its hopitality and financial support during a visit where part of this work was carried out.

References

[1] R. J. Adler: *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes.* Institute of Mathematical Statistics Lecture Notes-Monographs Series, Vol 12, 1990.

[2] R. Carmona, D. Nualart: *Random nonlinear wave equations: smoothness of the solutions.* Probab. Theory Related Fields 79, 469-508 (1988).

[3] R.C. Dalang, N.E. Frangos: *The stochastic wave equation in two spatial dimensions.* Annals of Probab. 26, 1, 187-212, 1998.

[4] R.C. Dalang: *Extending the martingale measure stochastic integral with applications to spatially homogeneous spde's.* Electronic J. of Probability, Vol 4, 1999.

[5] R.C. Dalang, C. Mueller: *Some non-linear SPDE's that are second order in time.* Electronic J. of Probability, Vol 8, 1, 1-21, 2003.

[6] R.C. Dalang, M. Sanz-Solé: *Hölder-Sobolev properties of the solution of the stochastic wave equation in dimension three.* In preparation.

[7] W.F. Donoghue: *Distributions and Fourier transforms.* Academic Press, New York, 1969.

[8] C. Gasquet, P. Witomski: *Fourier Analysis and Applications.* Texts in Applied Mathematics 30. Springer Verlag, 1999.

[9] L. Hörmander: *Lectures on Nonlinear Hyperbolic Differential Equations*. Springer Verlag, 1997.

[10] A. Karkzewska, J. Zabczyk: "Stochastic PDE's with function-valued solutions", in Clément Ph., den Hollander F., van Neerven J. and de Pagter B. (Eds), "Infinite-dimensional stochastic analysis", Proceedings of the Colloquium of the Royal Netherlands Academy of Arts and Sciences, 1999, Amsterdam.

[11] N.V. Krylov: *An analytic approach to spde's*. In: *Stochastic Partial Differential Equations: Six Perspectives* (R.A. Carmona, B. Rozovskii, Eds.), pp. 185-242. Mathematical Surveys and Monographs, Vol 64, American Mathematical Society, 1999.

[12] M. Métivier: *Semimartingales, a Course on Stochastic Processes*. de Gruyter Studies in Mathematics 2. Walter de Gruyter, 1982.

[13] A. Millet, M. Sanz-Solé: *A stochastic wave equation in two space dimensions: smoothness of the law*. Annals of Probab. 27, 803-844, 1999.

[14] S. Peszat, J. Zabczyk: *Nonlinear stochastic wave and heat equations.* Probab. Theory Related Fields, 116, 421-443, 2000.

[15] D. Revuz, M. Yor: *Continuous martingales and Brownian motion.* Third edition. Springer-Verlag, Berlin, 1999.

[16] M. Sanz-Solé, M. Sarrà: *Hölder continuity for the stochastic heat equation with spatially correlated noise*. In: *Stochastic analysis, random fields and applications* (R.C. Dalang, M. Dozzi , F. Russo Eds.), pp. 259-268, Progress in Probability 52, Birkhäuser, Basel, 2002.

[17] M. Sanz-Solé: *A course on Malliavin Calculus with Applications to Stochastic Partial Differential Equations*. Lecture Notes Vol. 2. Institut de Matemàtica. Universitat de Barcelona, 2004.

[18] L. Schwartz: *Théorie des distributions*. Hermann, Paris, 1966.

[19] N. Shimakura: *Partial differential operators of elliptic type*. Translations of Mathematical Monographs, 99. American Mathematical Society, 1992.

[20] C. D. Sogge: *Lectures on Nonlinear Wave Equations*. Monographs in Analysis, Vol II. International Press, 1995.

[21] F. Treves: *Basic Linear Partial Differential Equations.* Academic Press, 1975.

[22] J.B. Walsh: *An introduction to stochastic partial differential equations*, École d'été de Probabilités de Saint Flour XIV, Lecture Notes in Mathematics, Vol. 1180, Springer Verlag, 1986.